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Existence Theorems for Nonlinear Noncoercive Operator Equations

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INTRODUCTION

In this paper we study existence problems for equations of the form

$$f \in Au + Bu \tag{1}$$

in a real reflexive Banach space E. Here A is a subset of $E \times E^*$ and B is the subdifferential of a lower-semicontinuous convex function φ of E into $]-\infty, +\infty]$. There will be no further restrictions on the map φ. As for the operator A, we suppose that

(i) $\forall x \in \operatorname{Dom}(\varphi)$, Ax is a nonempty bounded closed convex subset of E^*; and

(ii) for each $[x_1, w_1]$ and $[x_2, w_2]$ in A

$$\langle w_1 - w_2, x_1 - x_2 \rangle + \lambda(x_1, x_2) \geqslant \beta(x_1 - x_2)$$

with λ a function of $E \times E$ into $]-\infty, +\infty[$ and β a lower-semicontinuous convex function of E into $]-\infty, +\infty[$.

In the recent years an extensive literature has arisen on the question of existence of solutions to equations of type (1), A being a linear operator with a finite dimensional kernel, the partial inverse of A being compact, and B being a nonlinear operator (see [9]). The case when both A and B are nonlinear and monotone has been studied by, among others, Brézis [3–5], Brézis and Haraux [8], Browder [11, 12], Browder and Hess [13], Calvert and Gupta [14], Gupta and Hess [17], and Reich [19].

Their method of proof of the solvability of (1) begins in a rather classical

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manner. With the aid of results for maximal monotone operators and degree theory they first solve the perturbed problem

$$f \in Au_\varepsilon + Bu_\varepsilon + \varepsilon J(u_\varepsilon) \tag{2}$$

for each $\varepsilon > 0$. Then, with the aid of the uniform boundedness principle they conclude the existence of a constant C, independent of ε, such that $\|u_\varepsilon\| \leqslant C$.

The abstract results obtained by this method take the form

$$R(A+B) \simeq R(A) + R(B),$$

or more generally,

$$R(A+B) \simeq R(A) + \operatorname{cov} R(B),$$

where $\operatorname{cov} K$ denotes the convex hull of K and $U \simeq V$ means the sets U and V have the same interiors and the same closures.

Here the question of the existence of a solution to Eq. (1), or more generally to

$$Cu \cap (Au + Bu) \neq \varnothing \tag{1'}$$

is reduced, with the aid of the minimax theorem, to the study of the variational problem

(VP): find $u \in \mathrm{Dom}(\varphi)$ such that

$$\inf_{(f,g)\in Cu\times Au} \langle f-g, v-u\rangle + \varphi(u) - \varphi(v) \leqslant 0 \quad \text{for all } v \in \mathrm{Dom}(\varphi).$$

The results obtained by this method do not depend on assumptions such as coerciveness, semi-coerciveness, asymptotic oddness, or maximal monotonicity of $A+B$, a fact which promises a considerably broader range of possible applications.

0. Preliminary Remarks

In this and the following sections, unless otherwise specified, E will denote a real reflexive Banach space and E^* will denote its dual. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x^*, x\rangle$. The norm of E will be denoted by $\|\cdot\|$ and the norm of E^* by $\|\cdot\|_*$.

We shall use the symbol "lim" or $\to$ to indicate strong convergence and "w-lim" or $\rightharpoonup$ for the convergence in $E_w = (E; \sigma(E, E^*))$. We shall also use the same notation to denote convergence in E^* and $E^*_w = (E^*; \sigma(E^*, E))$, respectively.

A subset M of $E \times E^*$ is said to be *monotone* (resp. *strongly monotone*) if for each pair $[x_i, y_i] \in M$, $i = 1, 2$, we have

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant 0$$

(resp. $\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant \|x_1 - x_2\|^2$). A monotone set is said to be *maximal monotone* if it is not properly contained in any other monotone set. The following notation is standard: for $A \subseteq E \times E^*$ we let

$$Ax = \{ y : [x, y] \in A \},$$

$$D(A) = \{x : Ax \neq \varnothing\}, \qquad R(A) = \bigcup \{Ax : x \in D(A)\}.$$

If $\lambda \in \mathbb{R}$ and B is a subset of $E \times E^*$, we also define

$$\lambda A = \{ [x, \lambda y] : [x, y] \in A \}$$

and

$$A + B = \{ [x, y + z] : [x, y] \in A, [x, z] \in B \}.$$

With each $x \in E$ we associate the set

$$J(x) = \{x^* \in E^* : \langle x^*, x \rangle = \|x\|^2 = \|x^*\|_*^2 \}.$$

Using the Hahn–Banach theorem, it is immediately clear that $J(x) \neq \varnothing$ for any $x \in E$. The set J ($\subseteq E \times E^*$) previously defined will be referred to as the (normalized) *Duality map* of E. If E and E^* are strictly convex, then J is a single-valued function defined on all of E.

DEFINITION 0.1. The subset A of $E \times E^*$ is said to be *upper semicontinuous* (as a map of $D(A) \subset E_w$ into $2^{E_w^*}$) if, for each $x_0 \in D(A)$ and each open set G in E_w^* containing Ax_0, there is a neighborhood U of x_0 in E_w such that $Ax \subseteq G$ for all $x \in U$.

A is said to be *finitely upper semicontinuous* if it is upper semicontinuous as a map of $F \cap D(A)$ into $2^{E_w^*}$, for each finite-dimensional subspace F of E.

The following proposition will be used latter:

PROPOSITION 0.2. *Let A be an upper semicontinuouos subset of $E \times E^*$, Suppose that for each $x \in D(A)$, Ax is a convex compact subset of E_w^*. If $x_n \rightharpoonup x_0$ in $D(A)$ and $x_n^* \in Ax_n$, then*

(i) $\sup_n \|x_n^*\|_* < +\infty$,

(ii) $\bigcap_{k=1}^{\infty} \overline{\operatorname{cov}}\{x_n^* : n \geqslant k\} \subseteq Ax_0$.

Remark. It is now clear that under the assumptions of the proposition above one can conclude that the set

$$\omega_W(\{x_n^*\}) = \{y^* \in E^* : y^* = w - \lim x_{n_k}^*, \\ \text{for some subsequence } \{x_{n_k}^*\} \text{ of } \{x_n^*\}\}$$

is nonempty and moreover $\omega_W(\{x_n^*\}) \subseteq Ax_0$.

Proof of Proposition 0.2 [21]. Assume $y \in E$. In view of the assumptions on A and the continuity (from E_w^* into $[0, +\infty[$) of

$$\psi: z^* \mapsto |\langle z^*, y\rangle|,$$

we can conclude that $Ax_0 \subseteq \psi^{-1}([0, \lambda[)$ for some $\lambda > 0$. Because of the upper semicontinuity of A at x_0, for all sufficiently large n we have

$$x_n^* \in Ax_n \subseteq \psi^{-1}([0, \lambda[);$$

hence $|\langle x_n^*, y\rangle| \leqslant \lambda(y)$ for all n. The uniform boundedness principle can now be applied to conclude that $\sup_n \|x_n^*\|_* < +\infty$. We now prove (ii). Let $z^* \notin Ax_0$; in view of the compactness of Ax_0 there exists $y \in E$ such that

$$\langle z^*, y\rangle > \langle b^*, y\rangle + \varepsilon$$

for each $b^* \in Ax_0$. Let V be an open set in E_w^*, with $Ax_0 \subseteq V$, such that

$$\langle z^*, y\rangle > \langle c^*, y\rangle + \varepsilon$$

for all $c^* \in V$. Again, by the upper semicontinuity of A at x_0, there exists an open set W (in E_w), $x_0 \in W$, for which $\langle z^*, y\rangle > \langle d^*, y\rangle + \varepsilon$ for all $d^* \in \bigcup_{x \in W} Ax$. Since $x_n \rightharpoonup x_0$ and $x_n^* \in Ax_n$,

$$\langle z^*, y\rangle > \langle x_n^*, y\rangle + \varepsilon$$

for all $n \geqslant n_0$. Therefore

$$\langle z^*, y\rangle \geqslant \langle w^*, y\rangle + \varepsilon$$

for each $w^* \in \overline{\text{cov}}\{x_n^* : n \geqslant n_0\}$, and $z^* \notin \bigcap_{k=1}^{\infty} \overline{\text{cov}}\{x_n^* : n \geqslant k\}$. ∎

Following Browder and Hess [13], a subset A of $E \times E^*$ is said to be *quasi-bounded* if for each $M > 0$ there exists $k(M) > 0$ such that whenever $[u, w] \in A$ and $\langle w, u\rangle \leqslant M\|u\|$, $\|u\| \leqslant M$, then $\|w\|_* \leqslant k(M)$.

A subset C of E is said to be *locally convex* if for each $x, y \in C$ there exists $t = t(x, y) \in]0, 1[$ such that

$$x(s) \equiv (1 - s)x + sy \in C$$

for each s, $0 \leqslant s < t$.

DEFINITION 0.3. A subset M of $E \times E^*$ is said to be *hemicontinuous on* C, $C \subseteq E$, if

(i) $C \cap D(A)$ is locally convex; and

(ii) for each $x, y \in C \cap D(A)$

$$\inf_{w \in Ax} \langle w, z \rangle \leqslant \overline{\lim_{t \to 0+}} \{ \sup_{w \in Ax(t)} \langle w, z \rangle \}$$

for each $z \in E$.

The following proposition (due to Brézis *et al.* [10]) is going to play a important role in the development of this paper:

PROPOSITION 0.4. *Let E be a Hausdorff topological vector space and X an arbitrary set in E. To each $x \in X$ let a set $F(x)$ in E be given satisfying*

(1) *$\overline{F(x_0)} = L$ is compact for some $x_0 \in X$.*

(2) *The convex hull of every finite subset $\{x_1, \ldots, x_n\}$ of X is contained in the corresponding union $\bigcup_{i=1}^n F(x_i)$.*

(3) *For every $x \in X$, the intersection of $F(x)$ with any finite-dimensional subspace is closed.*

(4) *For every convex subset D of E we have*

$$\overline{\left(\bigcap_{x \in X \cap D} F(x) \right)} \cap D = \left(\bigcap_{x \in X \cap D} F(x) \right) \cap D.$$

Then $\bigcap_{x \in X} F(x) \neq \varnothing$.

We close this section with a survey of basic convex analysis.

A function φ of E into $[-\infty, +\infty]$ is called *convex* if the inequality

$$\varphi(\lambda x + (1 - \lambda) y) \leqslant \lambda \varphi(x) + (1 - \lambda) \varphi(y)$$

holds for every $\lambda \in [0, 1]$ and all $x, y \in E$ such that the right-hand side is well defined. The function φ is called *concave* if $-\varphi$ is convex. The function φ is said to be *proper* if $\varphi(x) > -\infty$ for every $x \in E$ and if φ is not the constant $+\infty$.

DEFINITION 0.5. The function φ: $E \to [-\infty, +\infty]$ is called *lower semicontinuous* at x_0 if

$$\varphi(x_0) \leqslant \underline{\lim}_{x \to x_0} \varphi(x).$$

PROPOSITION 0.6. *Let* φ: $E \to [-\infty, +\infty]$ *be a convex and lower semicontinuous function. Then*

(1) *if* φ *assumes the value* $-\infty$, *it cannot take any finite value*;

(2) *the function* φ *is lower-semicontinuous with respect to the weak topology on* E;

(3) *the function* φ *is bounded from below by an affine continuous function.*

For the proof of this proposition see, e.g., [2]. Associated to any function φ: $E \to [-\infty, +\infty]$ we have its *conjugate map* φ^*: $E^* \to [-\infty, +\infty]$ defined by

$$\varphi^*(x^*) = \sup\{\langle x^*, x\rangle - \varphi(x) : x \in F\}.$$

If we define $\operatorname{Dom}(\varphi) \subseteq E$ by $\operatorname{Dom}(\varphi) = \{x \in E: \varphi(x) < +\infty\}$, then

$$\varphi^*(x^*) = \sup\{\langle x^*, x\rangle - \varphi(x): x \in \operatorname{Dom}(\varphi)\}.$$

The set $\operatorname{Dom}(\varphi)$ will be referred as the *effective domain* of φ.

DEFINITION 0.6. Let φ: $E \to [-\infty, +\infty]$ be a proper function. We say that the function φ is *subdifferentiable* at the point $u \in \operatorname{Dom}(\varphi)$ if its *subdifferential*

$$\partial\varphi(u) \equiv \{y^* \in E^*: \langle y^*, v-u\rangle \leqslant \varphi(v) - \varphi(u), \forall v \in E\}$$

is nonempty.

It is immediately obvious that $\partial\varphi$ ($\subseteq E \times E^*$) is monotone. In this context we have that if φ is assumed to be lower semicontinuous, proper, and convex on E, then $\partial\varphi$ is maximal monotone and it domain $D(\partial\varphi)$ is a dense subset of $\operatorname{Dom}(\varphi)$.

PROPOSITION 0.7. *Let* φ *be a lower semicontinuous proper convex function of* E *into* $]-\infty, +\infty]$. *Then*

(1) *function* φ^* *is covex, lower-semicontinuous, and proper on the dual space* E^*, *and*

(2) *for each* $x \in E$ *and* $y^* \in E^*$

$$\langle y^*, x\rangle \leqslant \varphi(x) + \varphi^*(y^*);$$

moreover,

$$\langle y^*, x\rangle = \varphi(x) + \varphi^*(y^*) \qquad \textit{if and only if} \qquad y^* \in \partial\varphi(x).$$

The proof of this proposition may be found in [2].

1. Main Results

As in the previous section, E will denote a real reflexive Banach space with topological dual E^*; $E_w^* = (E^*, \sigma(E^*, E))$.

Let φ be a proper lower semicontinuous convex function of E into $]-\infty, +\infty]$ and A a subset of $E \times E^*$. We shall say that the pair (φ, A) is ***admissible*** if the following conditions are satisfied:

$$\begin{array}{ll}\text{(i)} & \mathrm{Dom}(\varphi) \subseteq D(A); \\ \text{(ii)} & \text{For every } x \in D(A),\ Ax \text{ is convex and compact in } E_w^*.\end{array} \tag{A}$$

Definition 1.1. Let $\lambda\colon E \times E \to \mathbb{R}$ and $\beta\colon E \to \mathbb{R}$ be two functions with β convex and lower semicontinuous. Let A be a subset of $E \times E^*$. Then A is said to be of the class $M(\lambda, \beta)$ if

$$\langle w_1 - w_2, x_1 - x_2 \rangle + \lambda(x_1, x_2) \geqslant \beta(x_1 - x_2) \tag{B}$$

for each $[x_1, w_1]$ and $[x_2, w_2]$ in A. We note that in the case in which A is monotone, λ and β can be taken to be zero.

Proposition 1.2. *Let* (φ, A) *and* (φ, B) *be two admissible pairs. If* $u \in \mathrm{Dom}(\varphi)$ *is such that*

$$Bu \cap (Au + \partial\varphi(u)) \neq \varnothing, \tag{P}$$

then

$$0 \in \mathrm{Dom}(\varphi^*) + (A - B)(\mathrm{Dom}(\varphi)). \tag{Q}$$

Proof. Let $z \in Bu \cap (Au + \partial\varphi(u))$; then $z \in p + \partial\varphi(u)$ with $p \in Au$. Thus

$$q = z - p \in \partial\varphi(u) \qquad \text{and} \qquad u \in \partial\varphi^*(q).$$

From this inclusion we conclude that $q \in \mathrm{Dom}(\varphi^*)$ and $z = q + p \in \mathrm{Dom}(\varphi^*) + p$. Therefore

$$0 \in \mathrm{Dom}(\varphi^*) + p - z \subseteq \mathrm{Dom}(\varphi^*) + (A - B)(\mathrm{Dom}(\varphi)). \qquad \blacksquare$$

The results in the present section will show that under suitable conditions on the subsets A and B, condition (Q) is almost sufficient for the existence of a solution to (P).

In our discussion we shall use the finite topologies τ. A point x_0 of a set K lies in the τ-interior of K if and only if for each z in the containing space there exists $\delta(z) > 0$ such that for $0 \leqslant |r| < \delta(z)$, $x_0 + rz$ lies in K.

Lemma 1.3 will be of crucial importance to the subsequent development of this paper. The lemma as stated here is a slight modification of a similar one used by Aubin [1]. Although our poof does not differ from his, we shall give it here for the sake of completeness.

LEMMA 1.3. *Let E be an arbitrary Banach space and assume that both (φ, A) and (φ, B) are admissible pairs. Then the following two statements are equivalent*:

(1) $\exists \bar{u} \in \mathrm{Dom}(\varphi)$: $B\bar{u} \cap (A\bar{u} + \partial\varphi(\bar{u})) \neq \varnothing$;

(2) $\exists \bar{u} \in \mathrm{Dom}(\varphi)$: $\inf_{(f,g)\in B\bar{u}\times A\bar{u}} \langle f - g, v - \bar{u}\rangle + \varphi(\bar{u}) - \varphi(v) \leqslant 0$ $\forall v \in \mathrm{Dom}(\varphi)$.

Proof. The implication $(1) \Rightarrow (2)$ is obvious. Now, let $\bar{u}$ be a solution of (2) and $\mathscr{L}$ the mapping of $B\bar{u} \times A\bar{u} \times \mathrm{Dom}(\varphi)$ into $]-\infty, +\infty]$ defined by

$$\mathscr{L}(f, g, v) = \langle f - g, v - \bar{u}\rangle + \varphi(\bar{u}) - \varphi(v).$$

Clearly,

$$\inf_{(f,g)\in B\bar{u}\times A\bar{u}} \mathscr{L}(f, g, v) \leqslant 0$$

for each $v \in \mathrm{Dom}(\varphi)$. Therefore

$$\sup_{v\in\mathrm{Dom}(\varphi)} \{ \inf_{(f,g)\in B\bar{u}\times A\bar{u}} \mathscr{L}(f, g, v)\} \leqslant 0.$$

By the minimax theorem [15, p. 175] we can conclude that

$$\inf_{(f,g)\in B\bar{u}\times A\bar{u}} \{ \sup_{v\in\mathrm{Dom}(\varphi)} \mathscr{L}(f, g, v)\} \leqslant 0.$$

But

$$\sup_{v\in\mathrm{Dom}(\varphi)} \mathscr{L}(f, g, v) = \varphi^*(f - g) + \varphi(\bar{u}) - \langle f - g, \bar{u}\rangle$$

and

$$\inf_{(f,g)\in B\bar{u}\times A\bar{u}} \{\varphi^*(f - g) + \varphi(\bar{u}) - \langle f - g, \bar{u}\rangle\} \leqslant 0.$$

From Fenchel's inequality (Proposition 0.7(2)) we conclude that

$$\inf_{(f,g)\in B\bar{u}\times A\bar{u}} \{\varphi^*(f - g) + \varphi(\bar{u}) - \langle f - g, \bar{u}\rangle\} = 0.$$

Finally, assumption (A) and the lower semicontinuity of φ^* allow us to conclude the existence of $(\bar{f}, \bar{g}) \in B\bar{u} \times A\bar{u}$ for which

$$\varphi^*(\bar{f} - \bar{g}) + \varphi(\bar{u}) - \langle \bar{f} - \bar{g}, \bar{u}\rangle = 0.$$

This is equivalent to saying that $\bar{f} - \bar{g} \in \partial\varphi(\bar{u})$, i.e.,

$$B\bar{u} \cap (A\bar{u} + \partial\varphi(\bar{u})) \neq \varnothing. \quad \blacksquare$$

THEOREM 1.4. *Let E be a real reflexive Banach space and (φ, A) an admissible pair with A lying in the class $M(\lambda, \beta)$. If*

(1) $\lambda(x, y) \leqslant \beta(0)$ *for all* $x, y \in E$,

(2) $\tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\} \neq \varnothing$, *and either*

(3) *A is finitely upper semicontinuous, or*

(4) *A is hemicontinuous on* $\mathrm{Dom}(\varphi)$,

then

$$\tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\} \subseteq R(A + \partial\varphi).$$

COROLLARY 1.5. *Let E be a real reflexive Banach space and (φ, A) an admissible pair satisfying the assumptions of Theorem 1.4. If we also have*

(5) $\underline{\lim}_{\|x\| \to \infty}(\varphi(x)/\|x\|) = +\infty$,

then $R(A + \partial\varphi) = E^*$.

Proof. Let $x^* \in E^*$. From (5) there exists $R > 0$ such that

$$\varphi(x) \geqslant \|x^*\|_* \, \|x\|$$

for all x, $\|x\| \geqslant R$. Hence,

$$\langle x^*, x \rangle - \varphi(x) \leqslant \|x^*\|_* \{\|x\| - \|x\|\} \leqslant 0$$

if $\|x\| \geqslant R$. Since φ is lower semicontinuous and E is reflexive, we can conclude that

$$\varphi^*(x^*) = \sup_{x \in E} \{\langle x^*, x \rangle - \varphi(x)\} < +\infty.$$

Therefore, $\mathrm{Dom}(\varphi^*) = E^*$ and by Theorem 1.4

$$E^* \subseteq R(A + \partial\varphi). \quad \blacksquare$$

COROLLARY 1.6. *Let E be a real reflexive Banach space and (φ, A) an admissible pair satisfying the assumptions of Theorem 1.4. If $D(A) = E$, then for each $\varepsilon > 0$*

$$R(A + \varepsilon J) = E^*.$$

Proof. Let $\varepsilon > 0$ and $\varphi_\varepsilon: E \to [0, +\infty[$ be the map defined by

$$\varphi_\varepsilon(x) = (\varepsilon/2)\,\|x\|^2.$$

Clearly, $\mathrm{Dom}(\varphi_\varepsilon^*) = E^*$ and by Theorem 1.4 we can conclude that $E^* \subseteq R(A + \partial\varphi_\varepsilon)$. Since $\partial\varphi_\varepsilon = \varepsilon J$, the proof is completed. ∎

Remark. We remark in passing that in the case in which $[0, 0] \in A$ and all the assumptions of Corollary 1.6 are satisfied it follows that

$$\partial\beta(0) \subseteq R(A).$$

COROLLARY 1.7. *Let E be a real reflexive Banach space and (φ, A) an admissible pair satisfying the assumptions of Theorem* 1.4. *Assume $\partial\varphi$ is such that*

$$\forall f \in E^*, \quad \exists a \in E: \varlimsup_{\substack{[z,h] \in \partial\varphi \\ \|z\| \to +\infty}} \frac{\langle h - f, a - z\rangle}{\|z\|} < 0. \tag{$\ast\atop\ast$}$$

Then $R(A + \partial\varphi) = E^*$.

Proof. To prove Corollary 1.7 we first observe that if $f \in E^*$ and $\varepsilon > 0$, then $f = w_\varepsilon + \varepsilon J(u_\varepsilon)$ for some $u_\varepsilon \in D(\partial\varphi)$. Because of $({\ast\atop\ast})$, $\{\|u_\varepsilon\|: \varepsilon > 0\}$ is bounded. Hence $R(\partial\varphi) = E^*$. Finally, since $D(\partial\varphi^*) = R(\partial\varphi)$ and $D(\partial\varphi^*) \subset \mathrm{Dom}(\varphi^*)$, it follows that $\mathrm{Dom}(\varphi^*) = E^*$; hence $R(A + \partial\varphi) = E^*$. ∎

COROLLARY 1.8. *Let E be a real reflexive Banach space and let (φ, A) and (φ, B) be two admissible pairs with B strongly monotone and A lying in the class $M(\lambda, \beta)$. If*

(1) *$\lambda(x, y) \leqslant \beta(0)$ for all $x, y \in E$, and either*

(2) *$A + B$ is finitely upper semicontinuous, or*

(3) *$A + B$ is hemicontinuous on* $\mathrm{Dom}(\varphi)$,

then

$$R(A + B + \partial\varphi) = E^*.$$

Proof. Let $C = A + B$, and for each $u \in E$ define

$$\phi(u) = \|u\|^2 + \beta(u).$$

Clearly, $C \in M(\lambda, \phi)$ and

$$\lambda(u_1, u_2) \leqslant \phi(0)$$

for each u_1 and u_2 in E. In view of the lower semicontinuity of β, we can conclude the existence of $x^* \in E^*$ and $\alpha \in \mathbb{R}^+$ for which the inequality

$$\langle u^*, v\rangle - \phi(v) \leqslant \{\|u^*\|_* + \|x^*\|_*\}\, \|v\| - \|v\|^2 + \alpha$$

holds true, for each $v \in E$. Hence $\mathrm{Dom}(\phi^*) = E^*$ and the conclusion of Corollary 1.8 now follows froms from Theorem 1.4. ∎

DEFINITION 1.9. Let E be a real reflexive Banach space and A a subset of $E \times E^*$. Then A is said to be *pseudomonotone* if the following conditions hold:

(a) The set Au is nonempty, bounded, closed, and convex for all $u \in D(A)$.

(b) The set A is finitely upper semicontinuous as a map of E into $2^{E^*_w}$.

(c) If $\{u_j\}$ is a sequence in $D(A)$ converging weakly to $u \in D(A)$ and if $w_j \in Au_j$ is such that $\overline{\lim}_{j\to\infty}\langle w_j, u_j - u\rangle \leqslant 0$, then to each element $v \in D(A)$ there exists $w(v) \in Au$ with the property

$$\underline{\lim}_{j\to\infty} \langle w_j, u_j - v\rangle \geqslant \langle w(v), u - v\rangle.$$

THEOREM 1.10. *Let E be a real reflexive Banach space and (φ, A) an admissible pair with A a pseudomonotone operator of the class $M(\lambda, \beta)$, $\lambda(x, y) = \omega(x)\,\mu(y)$. Suppose that*

(1) $\|x\|^{-1}\,\omega(x) \to 0$ *as* $\|x\| \to +\infty$, *and*

(2) $\tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\} \neq \varnothing$.

Moreover, suppose that one of the following three conditions holds:

(3) *There exists $x \in \mathrm{Dom}(\varphi)$ such that*

$$\inf\{\langle w, u - x\rangle \colon [u, w] \in A, \|u\| \leqslant M\} > -\infty \quad \textit{for each} \quad M > 0. \qquad (+)$$

(4) *λ maps bounded subsets of $E \times E$ into bounded sunsets of $\mathbb{R}$.*

(5) *$0 \in \mathrm{Dom}(\varphi)$ and A is quasi-bounded.*

Then

$$\tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\} \subseteq R(A + \partial\varphi).$$

Remark. Condition (+) is implied by condition (*) of [8]:

$$\forall f \in R(A) \quad \forall y \in D(A) \quad \sup_{[z,h]\in A} \langle h - f, y - z\rangle < +\infty.$$

COROLLARY 1.11. *Let E be a real reflexive Banach space and (φ, A) an admissible pair with* $\mathrm{Dom}(\varphi) = E$. *If A is maximal monotone and satisfies condition* (+), *then*

$$\tau\text{-int}(R(\partial\varphi) + \operatorname{cov} R(A)) \subseteq R(A + \partial\varphi).$$

Proof. Since A is maximal monotone and $D(A) = E$, it follows that A is pseudomonotone. Now the corollary is an immediate consequence of Theorem 1.10 and the fact that if $\beta \equiv 0$, then $\mathrm{Dom}(\beta^*) = \{0\}$. ▮

In the subsequent results the following class of operators will play an important role:

DEFINITION 1.12. Let E be a real reflexive Banach space, (φ, A) an admissible pair, f an element of E^*, and $L(f)$: $\mathrm{Dom}(\varphi) \times \mathrm{Dom}(\varphi) \to \,]-\infty, +\infty]$ the mapping defined by

$$L(f)(x, y) = \inf_{w \in Ax} \langle w, x - y \rangle - \langle f, x - y \rangle + \varphi(x) - \varphi(y).$$

The pair (φ, A) is *firmly admissible* if

$$\begin{aligned}&\text{for each closed bounded subset } K \text{ of } \mathrm{Dom}(\varphi)\\ &K(f, y) = K \cap \{x \in \mathrm{Dom}(\varphi) : L(f)(x, y) \leqslant 0\}\\ &\text{is } \sigma(E, E^*)\text{-closed for each } y \in K \text{ and each } f \in E^*.\end{aligned} \tag{F}$$

Remark. Let (φ, A) be an admissible pair with A satisfying the following *strong* pseudomonotonicity condition:

If $x_n \rightharpoonup x$ with $x_n, x \in \mathrm{Dom}(\varphi)$, then

$$\inf_{w \in Ax} \langle w, x - y \rangle \leqslant \varliminf_{n \to \infty} \inf_{w \in Ax_n} \langle w, x_n - y \rangle$$

for each $y \in \mathrm{Dom}(\varphi)$.

Then (φ, A) is firmly admissible.

Proposition 1.13 provides us with a class of mappings A for which (φ, A) is firmly admissible.

PROPOSITION 1.13. *Let E be a reflexive Banach space and (φ, A) an admissible pair with A upper semicontinuous as a map of $D(A) \subseteq E_w$ into $2^{E_w^*}$. Assume that for each $[u_n, w_n] \in A$, $u_n \rightharpoonup u$ with u_n, $u \in$* $\mathrm{Dom}(\varphi)$

$$\varliminf_{n \to +\infty} \langle w_n, u_n - u \rangle \geqslant 0. \tag{$_*{}^*{}_*$}$$

Then (φ, A) is firmly admissible.

Proof. Let $[u_n, w_n] \in A$ be such that

$$\langle w_n, u_n - y \rangle = \inf_{w \in A u_n} \langle w, u_n - y \rangle.$$

Assume that $u_n \rightharpoonup u$ with u_n, $u \in \mathrm{Dom}(\varphi)$. Then (by Proposition 0.2), with $w = w-\lim w_{n_k}$ and assuming that $L(f)(u_n, y) \leqslant 0$, we have that $[u, w] \in A$ and

$$\langle w, u - y \rangle - \langle f, u - y \rangle + \varphi(u) - \varphi(y) \leqslant \overline{\lim}_{n \to \infty} \langle w_{n_k}, u - u_{n_k} \rangle \leqslant 0.$$

Hence

$$L(f)(u, y) \leqslant 0. \quad \blacksquare$$

Note that if A is monotone or, more generally, if there exists a lower semicontinuous convex function $\beta: E \to]-\infty, +\infty]$, $\beta(0) \geqslant 0$, such that

$$\langle w_1 - w_2, u_1 - u_2 \rangle \geqslant \beta(u_1 - u_2)$$

for each $[u_1, w_1]$ and $[u_2, w_2]$ in A, then A satisfies $({}_*{}^*{}_*)$.

THEOREM 1.14. *Let E be a real reflexive Banach space and (φ, A) a firmly admissible pair with A lying in the class $M(\lambda, \beta)$ and $\lambda(x, y) = \omega(x)\mu(y)$. Suppose that $\|x\|^{-1}\omega(x) \to 0$ as $\|x\| \to +\infty$, and*

$$\tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\} \neq \varnothing,$$

If any of the following five conditions holds:

(1) *A maps bounded subsets of E into bounded subsets of E^*;*

(2) *φ maps bounded subsets of E into bounded subsets of $\mathbb{R}$;*

(3) *$0 \in \mathrm{Dom}(\varphi)$ and A is quasi-bounded;*

(4) *There exists $x \in \mathrm{Dom}(\varphi)$ such that*

$$\inf\{\langle w, u - x \rangle : [u, w] \in A, \|u\| \leqslant M\} > -\infty \quad \textit{for each} \quad M > 0; \quad (+)$$

(5) *λ maps bounded subsets of $E \times E$ into bounded subsets of $\mathbb{R}$;*

then

$$\tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\} \subseteq R(A + \partial\varphi).$$

We now consider the solvability of (P). To this purpose, let (φ, A) and

(φ, B) be two admissible pairs and $\mathscr{L}$: $\mathrm{Dom}(\varphi) \times \mathrm{Dom}(\varphi) \to]-\infty, \infty]$ the mapping defined by

$$\mathscr{L}(u, v) = \inf_{(f,g) \in Bu \times Au} \langle f - g, v - u \rangle + \varphi(u) - \varphi(v).$$

DEFINITION 1.15. The pairs (φ, A), (φ, B) are said to be *jointly firmly admissible* if

$$\begin{aligned}&\text{for each closed convex bounded subset } K \text{ of } \mathrm{Dom}(\varphi)\\ &K(v) \equiv K \cap \{u: \mathscr{L}(u, v) \leqslant 0\}\\ &\text{is } \sigma(E, E^*)\text{-closed for each } v \in K.\end{aligned} \tag{SF}$$

THEOREM 1.16. *Let E be a real reflexive Banach space and (φ, A), (φ, B) two jointly firmly admissible pairs with A lying in the class $M(\lambda, \beta)$ and B satisfying*

$$\langle w_1 - w_2, x_1 - x_2 \rangle - \langle w_2, x_2 \rangle \leqslant D(x_2) < +\infty \tag{N}$$

for each $[x_1, w_1]$ and $[x_2, w_2]$ in B. Suppose $\lambda(x, y) = \omega(x)\mu(y)$ with $\|x\|^{-1}\omega(x) \to 0$ as $\|x\| \to +\infty$ and $0 \in \tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + (A - B)(\mathrm{Dom}(\varphi))\}$.

Supose further that any one of the following three conditions holds:

(1) *$0 \in \mathrm{Dom}(\varphi)$ and A is quasi-bounded.*

(2) *There exists $x \in \mathrm{Dom}(\varphi)$ such that*

$$\inf\{\langle w, u - x \rangle: [u, w] \in A, \|u\| \leqslant M\} > -\infty \quad \textit{for each} \quad M > 0. \tag{+}$$

(3) *λ maps bounded subsets of $E \times E$ into bounded subsets of $\mathbb{R}$.*

Then there exists $u \in \mathrm{Dom}(\varphi)$ such that

$$Bu \cap (Au + \partial\varphi(u)) \neq \varnothing.$$

Remark. Condition (N) is satisfied if B is the constant operator

$$B = \{[u, f]: u \in E, f \in E^* \ (f \text{ fixed})\}.$$

2. PROOF OF THEOREM 1.4.

For $f \in E^*$ let $L(f)$: $\mathrm{Dom}(\varphi) \times \mathrm{Dom}(\varphi) \to]-\infty, +\infty]$ be the map defined by

$$L(f)(x, y) = \inf_{w \in Ax} \langle w, x - y \rangle - \langle f, x - y \rangle + \varphi(x) - \varphi(y).$$

LEMMA 2.1. *Let C be a convex $\sigma(E, E^*)$-compact subset of* Dom(φ). *Then there exists $x_C \in C$ such that*

$$L(f)(x, x_C) \geqslant 0 \qquad (\mathrm{VI}_C)$$

for all $x \in C$.

Proof of Lemma 2.1. For each $x \in C$ let

$$W(x) = \{y \in C: L(f)(x, y) \geqslant 0\}.$$

Now, for $y_n \in W(x)$ and $y \in C$ with $y_n \to y$, if we let $w_n \in Ax$ be such that

$$\langle w_n, x - y_n \rangle = \inf_{w \in Ax} \langle w, x - y_n \rangle$$

for each $n \in \mathbb{N}$, we can conclude that

$$\langle \bar{w}, x - y \rangle - \langle f, x - y \rangle + \varphi(x) - \varphi(y) \geqslant 0,$$

where $\bar{w} = w - \lim w_{n_k}$, for some subsequence $\{w_{n_k}\}$ of $\{w_n\}$. Since

$$\langle \bar{w}, x - y \rangle \leqslant \overline{\lim_{k \to \infty}} \langle w_{n_k}, x - y_{n_k} \rangle \leqslant \langle \tilde{w}, x - y \rangle$$

for all $\tilde{w} \in Ax$, we obtain

$$L(f)(x, y) \geqslant 0.$$

Therefore, $W(x)$ is strongly closed and it is not difficult to see that $W(x)$ is also convex, hence $\sigma(E, E^*)$-compact. It is also easy to see that conditions (3) and (4) of Proposition 0.4 are satisfied. We now claim that if $x_1, ..., x_n \in C$, then

$$\operatorname{cov}(x_1, ..., x_n) \subseteq \bigcup_{i=1}^{n} W(x_i).$$

Suppose otherwise. Then there exist real numbers $\alpha_1, ..., \alpha_n$, $0 \leqslant \alpha_i \leqslant 1$, $\sum_{i=1}^{n} \alpha_i = 1$ for which

$$z = \sum_{i=1}^{n} \alpha_i x_i \notin W(x_j), \qquad 1 \leqslant j \leqslant n,$$

i.e., $L(f)(x_j, z) < 0$ for $1 \leqslant j \leqslant n$. Let $w_i(z) \in Ax_i$ be such that

$$L(f)(x_i, z) = \langle w_i(z), x_i - z \rangle - \langle f, x_i - z \rangle + \varphi(x_i) - \varphi(z).$$

Then

$$\sum_{i=1}^{n} \alpha_i L(f)(x_i, z) = \sum_{i=1}^{n} \alpha_i \langle w_i(z), x_i - z\rangle + \sum_{i=1}^{n} \alpha_i \varphi(x_i) - \varphi(z) < 0,$$

but

$$\sum_{i=1}^{n} \alpha_i \varphi(x_i) - \varphi(z) \geqslant 0,$$

hence

$$\sum_{i=1}^{n} \alpha_i \langle w_i(z), x_i - z\rangle < 0.$$

By assumption $\langle w_i(z), x_i - z\rangle \geqslant \beta(x_i - z) - \lambda(x_i, z) + \langle w, x_i - z\rangle$, where $w \in Az$. So

$$\begin{aligned}\sum_{i=1}^{n} \alpha_i \langle w_i(z), x_i - z\rangle &\geqslant \sum_{i=1}^{n} \alpha_i \beta(x_i - z) - \sum_{i=1}^{n} \alpha_i \lambda(x_i, z)\\ &\geqslant \beta(0) - \sum_{i=1}^{n} \alpha_i \lambda(x_i, z) \geqslant 0.\end{aligned}$$

This contradiction proves our claim, and in this way we have proved that the family $\{W(x)\colon x \in C\}$ satisfies all the requirements of Proposition 0.4. Thus

$$\bigcap_{x \in C} W(x) \neq \varnothing,$$

i.e., there exists x_C in C such that $L(f)(x, x_C) \geqslant 0$ for all $x \in C$. ∎

LEMMA 2.2. *If all the assumptions of Theorem* 1.4 *are satisfied, then there exists a sequence* $\{C_n\}$ *of subsets of* $\mathrm{Dom}(\varphi)$ *and a sequence* $\{x_n\}$ *of elements of* $\mathrm{Dom}(\varphi)$ *such that*

(1) *for each* $n \in \mathbb{N}$, $x_n \in C_n$ *and*
(2) *for each* $x \in C_n$ *and each* $n \in \mathbb{N}$, $L(f)(x_n, x) \leqslant 0$.

Proof of Lemma 2.2. For each $n \in \mathbb{N}$, let

$$C_n = \{x \in \mathrm{Dom}(\varphi)\colon \|x\| \leqslant n, \varphi(x) \leqslant n\};$$

clearly, C_n is a closed convex bounded subset of $\mathrm{Dom}(\varphi)$ and $\mathrm{Dom}(\varphi) = \bigcup_{n=1}^{\infty} C_n$.

We shall prove that if condition (3) or (4) of Theorem 1.4 is satisfied, then

x_n can be defined to be x_{C_n}, where x_{C_n} is the solution to (VI_C) with $C = C_n$. Let $t \in]0, 1[$, $x \in C_n$, and $x_n(t) \equiv x_{C_n} - t(x_{C_n} - x) \in C_n$. Then

$$L(f)(x_n(t), x_{C_n}) \geqslant 0, \qquad 0 < t < 1,$$

i.e.,

$$\inf_{w \in Ax_n(t)} \langle w, x_n(t) - x_{C_n}\rangle - \langle f, x_n(t) - x_{C_n}\rangle + \varphi(x_n(t)) - \varphi(x_{C_n}) \geqslant 0$$

or

$$\sup_{w \in Ax_n(t)} \langle w, x_{C_n} - x\rangle - \langle f, x_{C_n} - x\rangle + \varphi(x_{C_n}) - \varphi(x) \leqslant 0.$$

Assume now that A satisfies condition (3) of Theorem 1.4. Let $w_n(t) \in Ax_n(t)$ be such that

$$\langle w_n(t), x_{C_n} - x\rangle = \sup_{w \in Ax_n(t)} \langle w, x_{C_n} - x\rangle;$$

the upper semicontinuity of A on $F = sp\{x_{C_n}, x\}$ in conjunction with assumption (A) will imply that $\{w_n(t)\colon 0 < t < 1\}$ is bounded, and if $w_n = w\text{-}\lim_{t_m \to 0} w_n(t_m)$, then $w_n \in Ax_{C_n}$. Moreover,

$$\langle w_n, x_{C_n} - x\rangle - \langle f, x_{C_n} - x\rangle + \varphi(x_{C_n}) - \varphi(x) \leqslant 0.$$

Thus $L(f)(x_{C_n}, x) \leqslant 0$ for each $x \in C_n$.

If A satisfies condition (4) of Theorem 1.4, then

$$\begin{aligned}\inf_{w \in Ax_{C_n}} \langle w, x_{C_n} - x\rangle &\leqslant \overline{\lim_{t \to 0^+}} \{\sup_{w \in Ax_n(t)} \langle w, x_{C_n} - x\rangle\} \\ &\leqslant \langle f, x_{C_n} - x\rangle - \varphi(x_{C_n}) + \varphi(x).\end{aligned}$$

Hence $L(f)(x_{C_n}, x) \leqslant 0$ for each $x \in C_n$.

We can now conclude that in both instances, if $x_n \equiv x_{C_n}$, then $L(f)(x_n, x) \leqslant 0$ *for all x in C_n*. ∎

LEMMA 2.3. *Let $x_n \in \mathrm{Dom}(\varphi)$ and $C_n \subseteq \mathrm{Dom}(\varphi)$ be such that*

(1) *for each $n \in \mathbb{N}$, $x_n \in C_n$,*

(2) *for each $n \in \mathbb{N}$, $C_n \subseteq C_{n+1}$, and*

(3) *for each $x \in C_n$, $L(f)(x_n, x) \leqslant 0$.*

If $f \in \tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\}$, then there exists $M \in]0, +\infty[$ such that $\|x_n\| \leqslant M$ for each $n \in \mathbb{N}$.

Proof of Lemma 2.3. Let $z \in E^*$; then there exists $\delta(z) > 0$ such that

$$f + \delta z \in \mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))$$

for each $0 \leqslant |\delta| < \delta(z)$.

Hence

$$f + \delta z = \sum_{i=1}^{m(z)} \alpha_j\{a + b + \omega_j\},$$

where $a \in \mathrm{Dom}(\beta^*)$, $b \in \mathrm{Dom}(\varphi^*)$, and $\omega_j \in Ay_j$, $y_j \in \mathrm{Dom}(\varphi)$, $j = 1,..., m(z)$. If $n(z)$ is the smallest integer for which

$$\{y_i,..., y_{m(z)}\} \subseteq C_{n(z)}\ (\subseteq C_{n(z)+1} \subseteq \cdots),$$

then, for each $n \geqslant n(z)$,

$$\begin{aligned}\delta\langle z, x_n\rangle = \sum_{j=1}^{m(z)} \alpha_j\{&\langle a, x_n - y_j\rangle + \langle b, x_n\rangle + \langle w_j, x_n - y_j\rangle - \langle f, x_n - y_j\rangle \\ &+ \langle a + w_j - f, y_j\rangle\} \\ \leqslant \sum_{j=1}^{m(z)} \alpha_j\{&\beta^*(a) + \beta(x_n - y_j) + \varphi^*(b) + \varphi(x_n) + \langle w_j, x_n - y_j\rangle \\ &- \langle f, x_n - y_j\rangle + \langle a + w_j - f, y_j\rangle\}.\end{aligned}$$

But

$$\langle w_j, x_n - y_j\rangle \leqslant \lambda(x_n, y_j) - \beta(x_n - y_j) + \langle w, x_n - y_j\rangle$$

for each $w \in Ax_n$ and each j. Hence

$$\langle w_j, x_n - y_j\rangle \leqslant \lambda(x_n, y_j) - \beta(x_n - y_j) + \inf_{w \in Ax_n} \langle w, x_n - y_j\rangle$$

and

$$\begin{aligned}\delta\langle z, x_n\rangle \leqslant \sum_{j=1}^{m(z)} \alpha_j \lambda(x_n, y_j) + \sum_{j=1}^{m(z)} \alpha_j L(f)(x_n, y_j) \\ + \sum_{j=1}^{m(z)} \alpha_j\{\varphi(y_j) + \beta^*(a) + \varphi^*(b) + \langle a + w_j - f, y_j\rangle\}.\end{aligned}$$

Therefore

$$\langle z, x_n\rangle \leqslant \omega(z)\{\beta(0) + 1\} \qquad \forall n \geqslant n(z),$$

where

$$\omega(z) = \left\{\delta^{-1}; \delta^{-1} \sum_{j=1}^{m(z)} \alpha_j\{\varphi(y_j) + \beta^*(a) + \varphi^*(b) \right.$$
$$\left. + \langle a + w_j - f, y_j\rangle \right\}.$$

By the uniform boundedness principle there exists $M \in]0, +\infty[$ such that $\|x_n\| \leqslant M$ for each $n \in \mathbb{N}$. ∎

Proof of Theorem 1.4. As in Lemma 2.2, we let

$$C_n = \{x \in \mathrm{Dom}(\varphi) : \|x\| \leqslant n, \varphi(x) \leqslant n\}.$$

Since E is reflexive, C_n is $\sigma(E, E^*)$-compact for each $n \in \mathbb{N}$. Lemma 2.2 implies the existence of $x_n \in C_n$ for which $L(f)(x_n, x) \leqslant 0$ for each $x \in C_n$. Because $f \in \tau\text{-int}\{\mathrm{Dom}(\varphi^*) + \mathrm{Dom}(\beta^*) + \mathrm{cov}\, A(\mathrm{Dom}(\varphi))\}$, it follows from Lemma 2.3 that the sequence $\{\|x_n\|\}$ is uniformly bounded.

Let $x \in \mathrm{Dom}(\varphi)$ be such that $L(f)(x_n, x) \leqslant 0$ for each $n \in \mathbb{N}$. Then (for some $w_n \in Ax_n$)

$$\begin{aligned} -\infty < \varliminf_{n\to\infty} \varphi(x_n) \leqslant \varlimsup_{n\to\infty} \varphi(x_n) &\leqslant \varlimsup_{n\to\infty} \{\varphi(x) + \langle f, x_n - x\rangle - \langle w_n, x_n - x\rangle\} \\ &\leqslant \varphi(x) + \lim_{n\to\infty} \langle f, x_n - x\rangle + \beta(0) \\ &\quad - \lim_{n\to\infty} \beta(x_n - x) < +\infty. \end{aligned}$$

Hence $\{[x_n, \varphi(x_n)] : n \in \mathbb{N}\}$ is bounded in $E \times \mathbb{R}$. To complete the proof of the theorem, we let

$$R = \max\{\sup_n \|x_n\|; \sup_n \varphi(x_n)\}$$

and $N = [2R + 1]$. Then, for each $y \in \mathrm{Dom}(\varphi)$ there exists $\lambda \in]0, 1[$ such that $v = (1 - \lambda)x_N + \lambda y \in C_N$. Hence

$$0 \geqslant L(f)(x_N, v) \geqslant (1 - \lambda)\, L(f)(x_N, x_N) + \lambda L(f)(x_N, y)$$

and $$L(f)(x_N, y) \leqslant 0 \qquad \text{for all} \quad y \in \mathrm{Dom}(\varphi).$$

The conclusion of the theorem is now a consequence of Lemma 1.3.

3. Proof of Theorem 1.10

As in [18], we shall denote by $E \oplus \mathbb{R}$ the space $E \times \mathbb{R}$, normed by

$$\|(x, \alpha)\| = (\|x\|^2 + |\alpha|^2)^{1/2}.$$

We identify the dual $(E \oplus \mathbb{R})^*$ of $E \oplus \mathbb{R}$ with $E^* \oplus \mathbb{R}$, the pairing between $(x, \alpha) \in E \oplus \mathbb{R}$ and $(y^*, \beta^*) \in E^* \oplus \mathbb{R}$ being

$$\langle\langle (y^*, \beta^*), (x, \alpha) \rangle\rangle = \langle y^*, x \rangle + \beta^* \alpha.$$

For any subset A of $E \times E^*$, we shall denote by $A \oplus 1$ the subset of $(E \oplus \mathbb{R}) \times (E^* \oplus \mathbb{R})$ defined by

$$A \oplus 1(x, \alpha) = \{(w, 1) : w \in Ax\}.$$

Clearly, $A \oplus 1$ is pseudomonotone, provided A is such. Moreover,

$$\operatorname{epi}(\varphi) = \{(x, \alpha) : \alpha \geqslant \varphi(x)\} \subseteq D(A \oplus 1)$$

if $\operatorname{Dom}(\varphi) \subseteq D(A)$.

For $f \in E^*$ and K a subset of $D(A)$, we let $S(A, K)$ be the set of all elements u of E such that

(i) $\quad u \in K: \inf_{w \in Au} \langle w - f, u - v \rangle \leqslant 0, \qquad \forall v \in K.$

If $\operatorname{Dom}(\varphi) \subseteq D(A)$, we shall denote by $S(A, \varphi)$ the set of all elements u of $D(A)$ such that

(ii) $\quad u \in \operatorname{Dom}(\varphi): \inf_{w \in Au} \langle w - f, u - v \rangle + \varphi(u) - \varphi(v) \leqslant 0 \quad \forall v \in \operatorname{Dom}(\varphi).$

Then

$$(u, \alpha) \in S(A \oplus 1, \operatorname{epi}(\varphi))$$

if and only if

$$u \in S(A, \varphi) \qquad \text{and} \qquad \alpha = \varphi(u).$$

Lemma 3.1. *Let E be a real reflexive Banach space and (φ, A) an admissible pair with A a pseudomonotone operator of* $\operatorname{Dom}(\varphi)$ *into 2^{E^*}. If $R > 0$ and*

$$\operatorname{Dom}(\varphi, R) = \{x \in \operatorname{Dom}(\varphi) : \|x\| \leqslant R, \varphi(x) \leqslant R\} \neq \varnothing,$$

then

$$S(A \oplus 1, \operatorname{epi}(\varphi, R)) \neq \varnothing,$$

where $\operatorname{epi}(\varphi, R) = \{[x, \alpha] \in \operatorname{epi}(\varphi): x \in \operatorname{Dom}(\varphi, R),\ \alpha \leqslant R\}$.

Proof of Lemma 3.1. Because of [13, Theorem 15], we only need to verify that $\operatorname{epi}(\varphi, R)$ is a bounded subset of $E \oplus \mathbb{R}$. Suppose otherwise. Then $\|[x_n, \alpha_n]\| \to +\infty$ as $n \to +\infty$ for some sequence $\{[x_n, \alpha_n]\} \subseteq \operatorname{epi}(\varphi, R)$. Since $\|x_n\| \leqslant R$, and $\alpha_n \leqslant R$, it follows that $\alpha_n \to -\infty$ as $n \to +\infty$. Also, $\varphi(x_n) \leqslant \alpha_n$ for each $n \in \mathbb{N}$. Thus $\varphi(x_n) \to -\infty$ as $n \to +\infty$. The reflexivity of E and the lower-semicontinuity of φ will then lead us to a contradiction. ∎

Remark. We remark in passing that if $[x_R, \alpha_R]$ is in $S(A \oplus 1, \operatorname{epi}(\varphi, R))$, then so is $[x_R, \varphi(x_R)]$.

Throughout our arguments, we will make use of

LEMMA 3.2. *Let E be an arbitrary Banach space and ω: $E \to \mathbb{R}$ a function with $\|x\|^{-1}\omega(x) \to 0$ as $\|x\| \to +\infty$. Let $\{x_n\}$, $x_n \in E$, be such that for each $y \in E^*$ there exist a constant $c(y) \geqslant 0$ and an integer $n(y)$ for which $\langle y, x_n \rangle \leqslant c(y)\{\omega(x_n) + M(y)\}$, $M(y) < +\infty$ for each $n \geqslant n(y)$. Then $\{\|x_n\|\}$ is bounded.*

A similar lemma was proved by Browder [12]. Although our proof is the same as his, we shall give it here for the sake of completeness.

Proof of Lemma 3.2. Suppose otherwise. Then we may assume without loss of generality that $\|x_n\| \to +\infty$ and that $|\omega(x_n)| \to \infty$.

Let $v_n = |\omega(x_n)|^{-1} x_n$. Then $\|v_n\| \to +\infty$ while for each y in E^*,

$$\langle y, v_n \rangle \leqslant c(y)\{1 + (M(y)/|\omega(x_n)|)\}, \qquad n \geqslant n(y).$$

Hence

$$\langle y, v_n \rangle \leqslant D(y) < +\infty, \qquad \forall n \geqslant 1,$$

where

$$D(y) \equiv \max\{\max_{1 \leqslant i \leqslant n(y)} |\langle y, v_i \rangle|;\ \max_{i > n(y)} c(y)\{1 + (M(y)/|\omega(x_i)|)\}\}.$$

By the uniform boundedness principle $\{\|v_n\|\}$ is bounded. This contradiction proves the lemma. ∎

Proof of Theorem 1.10. As in Lemma 3.1, we let

$$\operatorname{Dom}(\varphi, R) = \{x \in \operatorname{Dom}(\varphi): \|x\| \leqslant R,\ \varphi(x) \leqslant R\}$$

and

$$\text{epi}(\varphi, R) = \{[x, \alpha] \in \text{epi}(\varphi) : x \in \text{Dom}(\varphi, R), \alpha \leqslant R\}.$$

Then $\text{epi}(\varphi) = \bigcup_{R>0} \text{epi}(\varphi, R)$, and for each $R \geqslant R_0$ there exists $[x_R, \alpha_R] \in \text{epi}(\varphi, R)$ such that $[x_R, \varphi(x_R)] \in S(A \oplus 1, \text{epi}(\varphi, R))$.

Since $\{[x, \varphi(x)] : x \in \text{Dom}(\varphi, R)\} \subseteq \text{epi}(\varphi, R)$, it follows that

$$\inf_{w \in A x_R} \langle w, x_R - y \rangle - \langle f, x_R - y \rangle + \varphi(x_R) - \varphi(y) \leqslant 0$$

for each $y \in \text{Dom}(\varphi, R)$.

As in Lemma 2.3, we can see that if

$$f \in \tau\text{-int}\{\text{Dom}(\varphi^*) + \text{Dom}(\beta^*) + \text{cov}\, A(\text{Dom}(\varphi))\},$$

then for each $z \in E^*$ there exists $\delta(z) > 0$ such that if $0 \leqslant |\delta| < \delta(z)$, then

$$f + \delta z \in \text{Dom}(\varphi^*) + \text{Dom}(\beta^*) + \text{cov}\, A(\text{Dom}(\varphi))$$

and

$$\delta\langle z, x_R \rangle \leqslant \sum_{j=1}^{m(z)} \alpha_j \mu(y_j)\, \omega(x_R) + M(z), \qquad R \geqslant R_0,$$

where

$$\delta z = \sum_{j=1}^{m(z)} \alpha_j \{a + b + \omega_j - f\}$$

with $a \in \text{Dom}(\beta^*)$, $b \in \text{Dom}(\varphi^*)$, $\omega_j \in A y_j$, $y_j \in \text{Dom}(\varphi)$, $j = 1, \dots, m(z)$. Here R_0 is the smallest real for which

$$\{y_1, \dots, y_{m(z)}\} \subseteq \text{Dom}(\varphi, R_0)\ (\subseteq \text{Dom}(\varphi, R_0 + \varepsilon), \varepsilon > 0).$$

Hence

$$\delta\langle z, x_R \rangle \leqslant \max_{1 \leqslant j \leqslant m(z)} |\mu(y_j)|\, |\omega(x_R)| + M(z),$$

and if we let

$$C(z) = \max\{\delta^{-1} \max_{1 \leqslant j \leqslant m(z)} |\mu(y_j)|; \delta^{-1} M(z)\},$$

then

$$\langle z, x_R \rangle \leqslant C(z)\{|\omega(x_R)| + 1\}$$

for each $R \geqslant R_0$. By Lemma 3.2, the sequence $\{\|x_R\| : R \geqslant R_0\}$ is bounded.

To conclude the proof of the theorem it suffices to show that $\{\varphi(x_R)$: $R \geqslant R_0\}$ is bounded. To this purpose we first assume that condition (3) is satisfied. Then

$$\varphi(x_R) \leqslant \varphi(x) + \langle f, x_R - x\rangle - \langle w_R, x_R - x\rangle$$

for some $x \in \bigcap_{R \geqslant R_0} \mathrm{Dom}(\varphi, R)$ and some $w_R \in Ax_R$. Since $\{\|x_R - x\|$: $R \geqslant R_0\}$ is bounded, condition (+) will then imply that

$$\langle w_R, x_R - x\rangle \geqslant -M > -\infty$$

for each $R \geqslant R_0$, and

$$\varphi(x_R) \leqslant R_0 + \|f\|_* \sup_{R \geqslant R_0} \|x_R - x\| + M < +\infty.$$

The reflexivity of E and the lower semicontinuity of φ will then imply that $\{\varphi(x_R)$: $R \geqslant R_0\}$ is bounded.

If (4) is satisfied, then

$$\varphi(x_R) \leqslant \varphi(x) + \langle f, x_R - x\rangle - \beta(x_R - x) + \lambda(x_R, x) - \langle w, x_R - x\rangle$$

for each $R \geqslant R_0$, $x \in \mathrm{Dom}(\varphi, R_0)$, and $w \in Ax$. Again, the lower semicontinuity of φ and β, the boundedness of λ, and the reflexivity of E will imply that $\{\varphi(x_R)$: $R \geqslant R_0\}$ is bounded.

Finally, if (5) is satisfied and if $\varphi(x_R) \to +\infty$ as $R \to +\infty$, then

$$\langle w_R, x_R\rangle \leqslant \varphi(0) - \varphi(x_R) + \langle f, x_R\rangle$$

for some $w_R \in Ax_R$ and each $R \geqslant R_0$. So, for each $R \geqslant R_1 \geqslant R_0$, $\langle w_R, x_R\rangle \leqslant \|f\|_* \|x_R\|$. Let $M_1 = \sup\{\|x_s\|$: $s \geqslant R_1\}$ and $M = \mathrm{Max}\{M_1, \|f\|_*\}$; then

$$\langle w_R, x_R\rangle \leqslant M \|x_R\| \qquad \text{and} \qquad \|x_R\| \leqslant M$$

for each $R \geqslant R_1$. By the quasi-boundedness of A we can conclude that

$$\|w_R\|_* \leqslant K(M) < +\infty$$

for each $R \geqslant R_1$. Hence

$$\varphi(x_R) \leqslant \varphi(0) + \langle f, x_R\rangle + \|w_R\|_* \|x_R\| < \infty.$$

This contradiction proves that $\{\varphi(x_R)$: $R \geqslant R_0\}$ is bounded above. Now, the boundedness of $\{\varphi(x_R)$: $R \geqslant R_0\}$ follows as in the previous cases. ∎

4. Proof of Theorem 1.14

For $f \in E^*$ let $L(f)$: $\mathrm{Dom}(\varphi) \times \mathrm{Dom}(\varphi) \to \,]-\infty, +\infty]$ be the map defined by

$$L(f)(x, y) = \inf_{w \in Ax} \langle w, x - y\rangle - \langle f, x - y\rangle + \varphi(x) - \varphi(y).$$

Lemma 4.1. *If E is supplied with its weak topology $\sigma(E, E^*)$, if K is a closed convex bounded subset of* $\mathrm{Dom}(\varphi)$, *and if (φ, A) is a firmly admissible pair, then there exists $x_K \in K$ such that*

$$L(f)(x_k, y) \leqslant 0 \tag{$\mathrm{VI_K}$}$$

for each $y \in K$.

Proof of Lemma 4.1 [20]. Suppose $(\mathrm{VI_K})$ does not hold. If for each $y \in K$ we set

$$K(y) = \{x \in K: L(f)(x, y) > 0\},$$

it follows that $K \backslash K(y) = \{x \in K\colon L(f)(x, y) \leqslant 0\}$ is $\sigma(E, E^*)$-closed. Since K is $\sigma(E, E^*)$-compact and $K = \bigcup_{y \in K} K(y)$, there exists a finite family $\{y_k,..., y_k\}$ such that $K = \bigcup_{i=1}^k K(y_i)$. Let $\{p_1,..., p_k\}$ be a partition of unity corresponding to this covering, i.e., each p_i is a continuous mapping of K into $[0, 1]$ which vanishes outside of $K(y_i)$, while $\sum_{i=1}^k p_i(x) = 1$ for all x in K.

Let p be the mapping defined by

$$p(x) = \sum_{i=1}^{k} p_i(x)\, y_i$$

for each $x \in K$. Then p is continuous, and by Brouwer's fixed point theorem there exists $\bar{x}$ in K such that

$$\bar{x} = p(\bar{x}) = \sum_{i=1}^{k} p_i(\bar{x})\, y_i.$$

Hence

$$0 \geqslant L(f)\left(\bar{x}, \sum_{i=1}^{k} p_i(\bar{x})\, y_i\right) \geqslant \sum_{i=1}^{k} p_i(\bar{x})\, L(f)(\bar{x}, y_i) > 0.$$

This contradiction proves the Lemma. ∎

Proof of Theorem 1.14. As in Lemma 3.1, we let

$$\mathrm{Dom}(\varphi, R) = \{w \in \mathrm{Dom}(\varphi) : \|x\| \leqslant R, \varphi(x) \leqslant R\}.$$

Since E is reflexive, $\mathrm{Dom}(\varphi, R)$ is $\sigma(E, E^*)$-compact, and by Lemma 4.1 there exists $x_R \in \mathrm{Dom}(\varphi, R)$ $(R \geqslant R_0)$ such that $L(f)(x_R, y) \leqslant 0$ for each $y \in \mathrm{Dom}(\varphi, R)$.

The proof of Theorem 1.14 now mimics that of Theorem 1.10 and it is omitted. ∎

5. Proof of Theorem 1.16

Let (φ, A) and (φ, B) be jointly firmly admissible and let $\mathscr{L}$: $\mathrm{Dom}(\varphi) \times \mathrm{Dom}(\varphi) \to]-\infty, +\infty]$ be the mapping defined by

$$\mathscr{L}(u, v) = \inf_{(f,g) \in Bu \times Au} \langle f - g, v - u\rangle + \varphi(u) - \varphi(v).$$

Lemma 5.1. *If E is supplied with its weak topology $\sigma(E, E^*)$, if K is a closed convex bounded subset of* $\mathrm{Dom}(\varphi)$, *and if (φ, A) and (φ, B) are jointly firmly admissible, then there exists $x_K \in K$ such that*

$$\mathscr{L}(x_K, y) \leqslant 0$$

for each $y \in K$.

Proof of Theorem 1.16. For each $R > 0$ we let

$$\mathrm{Dom}(\varphi, R) = \{x \in \mathrm{Dom}(\varphi) : \|x\| \leqslant R, \varphi(x) \leqslant R\}.$$

Since E is reflexive and $\mathrm{Dom}(\varphi, R)$ is closed, convex, and bounded, there exists $x_R \in \mathrm{Dom}(\varphi, R)$ such that $\mathscr{L}(x_R, y) \leqslant 0$ for each $y \in \mathrm{Dom}(\varphi, R)$.

By assumption, for each $p \in E^*$ there exists $\delta(p) > 0$ such that

$$\delta(p)\, p = r + q + f - g$$

with $r \in \mathrm{Dom}(\beta^*)$, $q \in \mathrm{Dom}(\varphi^*)$, $f \in Au$, and $g \in Bu$ for some $u \in \mathrm{Dom}(\varphi, R_0)$. For each $R \geqslant R_0$

$$\begin{aligned}\delta(p)\langle p, x_R\rangle &= \langle r, x_R - u\rangle + \langle q, x_R\rangle + \langle g, x_R - u\rangle - \langle g, x_R\rangle + \langle r + f, u\rangle \\ &\leqslant \beta^*(r) + \beta(x_R - u) + \varphi^*(q) + \varphi(x_R) - \langle w_R - f, x_R - u\rangle \\ &\quad + \langle w_R, x_R - u\rangle - (g, x_R\rangle + \langle f + r, u\rangle\end{aligned}$$

for some $w_R \in Ax_R$. Hence

$$\delta(p)\langle p, x_R\rangle \leqslant \{\langle \bar{w}_R - w_R, u - x_R\rangle + \varphi(x_R) - \varphi(u)\} + \lambda(x_R, u) \\ + \{\langle \bar{w}_R, x_R - u\rangle - \langle g, x_R\rangle\} + \varphi^*(q) + \beta^*(r) + \langle f + r, u\rangle$$

for each $\bar{w}_R \in Bx_R$ and each $w_R \in Ax_R$. So

$$\delta(p)\langle p, x_R\rangle \leqslant \mathscr{L}(x_R, u) + \lambda(x_R, u) + \{\langle \bar{w}_R - g, x_R - u\rangle - \langle g, u\rangle\} \\ + \varphi^*(q) + \beta^*(r) + \langle f + r, u\rangle.$$

Therefore

$$\delta(p)\langle p, x_R\rangle \leqslant \mu(u)\,\omega(x_R) + M(p), \qquad R \geqslant R_0,$$

where $M(p) < +\infty$. By Lemma 3.2 the sequence $\{\|x_R\|: R \geqslant R_0\}$ is bounded. As in Theorem 1.10 we can conclude that $\{\varphi(x_R): R \geqslant R_0\}$ is also bounded.

To complete the proof of the theorem we let

$$R = \max\{\sup \|x_R\|; \sup \varphi(x_R)\}$$

and $N = [2R + 1]$. Then for each $y \in \mathrm{Dom}(\varphi)$ there exists $\lambda \in]0, 1[$ such that

$$v = (1 - \lambda)x_N + \lambda y \in \mathrm{Dom}(\varphi, N).$$

Hence

$$0 \geqslant \mathscr{L}(x_N, v) \geqslant (1 - \lambda)\,\mathscr{L}(x_N, x_N) + \lambda\mathscr{L}(x_N, y)$$

and

$$\mathscr{L}(x_N, y) \leqslant 0$$

for all $y \in \mathrm{Dom}(\varphi)$.

The conclusion of the Theorem now follows as a consequence of Lemma 1.3. ∎

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